Properties of border states of transient chaos

A. Németh
Institute for Solid State Physics, Eötvös University, Múzeum körút 6-8, H-1088 Budapest, Hungary

P. Szépfalusy
Institute for Solid State Physics, Eötvös University, Múzeum körút 6-8, H-1088 Budapest, Hungary
and Research Institute for Solid State Physics, P.O. Box 49, H-1525 Budapest, Hungary
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Transient chaos with absolutely continuous conditionally invariant measure is studied, and a general solution of the Frobenius-Perron equation is presented for complete one-dimensional maps. In particular, properties of borderline situations exhibiting phase transitions within the framework of thermodynamical formalism are investigated.

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I. INTRODUCTION AND SUMMARY

It has become clear in the last few years that transient chaos plays an important role in nonlinear systems [1–6] (and references quoted therein). Transient chaos can occur, for instance, before the trajectory settles down in phase space on the attractor (which can be simple or strange as well). The length of the transient depends on the initial value and even for typical initial conditions the chaotic behavior of the transient can be often well observed over a considerable time interval. Besides detecting transient chaotic signals, transient chaos has come into the focus of interest recently because of its connections to chaotic scattering and deterministic diffusion.

Transient chaos is associated with an invariant object in phase space, called chaotic repeller (more precisely chaotic saddle), which is a Cantor-like set. All initial values lead to trajectories that escape from the neighborhood of the repeller with the exception of points on it. An important step in the development of the theory of transient chaos was the recognition that a so-called conditionally invariant measure can be defined by compensating for the escape on the average [7,8].

The purpose of the present paper is to give a general solution for the conditionally invariant measure in a class of complete one-dimensional (hereafter 1D) maps exhibiting transient chaos. Then we discuss, in particular, the properties of a critical state which is a generalization of the weak intermittent state of permanent to transient chaos.

We consider a map

$$x_{i+1} = f(x_i)\,, \quad x_i \in I_0, I_1 \qquad (1)$$

with one increasing and one decreasing monotonic branch, illustrated in Fig. 1. These branches map, respectively, the subintervals I_0 and I_1, to the interval I.

Map (1) is not defined for the points in the window (i.e., for the points in I outside the two subintervals I_0 and I_1). Such points escape from the interval in the next step and their fates are governed by some dynamics whose specification is not needed for our purposes. It is assumed that map (1) does not have a stable periodic orbit but can generate erratically-behaving long orbits. Since — except for initial points whose set is of zero Lebesgue measure — all the iterates finally escape, this type of chaos is called a transient one.

Let us define a bipartition with elements I_0, I_1, and

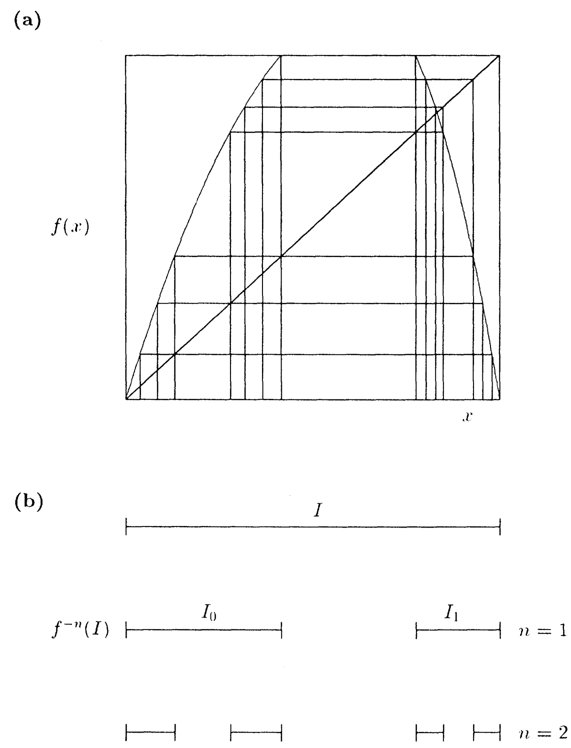

FIG. 1. 1D map generating transient chaos on unit interval I. Cylinder sets up to level 3 are shown in the lower part of the figure.

a corresponding symbolic dynamics $i = 0, 1$. The map f is complete in the sense that all the possible symbolic code combinations are allowed. Orbits of length n with different symbol sequences $(i_1, \ldots, i_n)$ start from the 2^n intervals (cylinder sets)

$$I_l^{(n)} = I_{i_1} \bigcap f^{-1}(I_{i_2}) \bigcap \cdots \bigcap f^{-n+1}(I_{i_n}) ,$$
$$l = \sum_{m=1}^{n} i_m 2^{m-1} , \quad n = 1, 2, \ldots . \tag{2}$$

Here $f^{-k}(I_j)$ stands for the points mapped to I_j by k applications of the map f. (Note that $I_i^{(1)} \equiv I_i$.) The cylinder sets (2) are shown in Fig. 1 up to level 3. In the limit $n \to \infty$ the cylinder sets I_l, $l = 0, \ldots, 2^n - 1$, approach a Cantor-set-like object, the chaotic repeller.

It is assumed that by compensating for the escape from the interval I, a smooth measure (absolutely continuous with respect to the Lebesgue measure) exists, whose density $P(x)$ satisfies the Frobenius-Perron equation [7,8]

$$P(x) = R \sum_{x_i \in f^{-1}(x)} \frac{P(x_i)}{|f'(x_i)|} . \tag{3}$$

The eigenvalue R is related to the escape rate κ by $\kappa = \ln R$. Here and in the following prime denotes the derivative. The measure is called conditionally invariant and plays an important role in the theory of transient chaos [3,5].

The measure of the cylinder sets of level n is given by

$$\mu_i^{(n)} = R^n \int_{I_i^{(n)}} P(x)\, dx , \quad i = 0, 1, \ldots, 2^n - 1 . \tag{4}$$

The prefactor R^n is needed to ensure normalization

$$\sum_{i=0}^{2^n-1} \mu_i^{(n)} = 1 . \tag{5}$$

In the limit $n \to \infty$ $\mu^{(n)}$ can be considered as the natural measure on the repeller [9,10].

We summarize below the main results of the paper along with its organization. In Sec. II we introduce two types of transformations of complete 1D maps, which leave the escape rate invariant. In particular, it is pointed out that starting from the tent map all the maps with a conditionally invariant absolute continuous measure (and with the same escape rate) can be reached by a combination of such transformations. Furthermore, since only one type of the transformations changes the statistical properties of the map, we can restrict ourselves to the application of this type in the rest of the paper. Section III is devoted to studying two limiting situations, where hyperbolicity becomes violated due to the appearance of infinite derivatives of the map either at the right or at the left end of the interval I_0. In the former case the Lyapunov exponent of the fixed point at the origin is equal to the escape rate. This situation is shown to lead to a similar type of phase transition, within the framework of the thermodynamic formalism, as the one occurring in permanent chaos when weak intermittency is present. There is, however, an important difference, supported also by numerical results, namely, that the transition is continuous for $\kappa > 0$, while it is known to be discontinuous at the limit of permanent chaos. Finally, a number of further phase transitionlike singularities will be pointed out and discussed.

II. CLASSIFICATION OF COMPLETE MAPS

One can specify a class of maps with the same escape rate. Obviously we can obtain such maps with the help of a smooth coordinate transformation, called conjugation, studied extensively in case of permanent chaos [11,12]. In the following the interval I will be taken, by appropriate rescaling, as $[0, 1]$; then the intervals I_0 and I_1 become $[0, x_{m_1}]$ and $[x_{m_2}, 1]$, respectively. Introducing the smooth monotonic function $u(x)$ with $u(0) = 0$, $u(1) = 1$, we get for the transformed map

$$g(x) = u\left(f(u^{-1}(x))\right) , \quad 0 \le x \le x_{m_1,g},$$
$$1 - x_{m_2,g} \le x \le 1, \quad x_{m_i,g} = u(x_{m_i,f}) \tag{6}$$

and for the transformed density

$$P_g(x) = \frac{P_f(u^{-1}(x))}{|u'(u^{-1}(x))|} . \tag{7}$$

One can easily convince oneself that $g(x)$ satisfies the Frobenius-Perron equation (3) with the same escape rate κ as $f(x)$. Furthermore each map is conjugated to a symmetric one, i.e., for which $f(x) = f(1-x)$ is fulfilled, and consequently without loss of generality we can restrict ourselves in the following to maps possessing this symmetry. The simplest such map is the piecewise linear map

$$f_0(x) = 2Rx, \quad 0 \le x \le \frac{1}{2R} ,$$
$$f_0(x) = 2R(1-x), \quad 1 - \frac{1}{2R} \le x \le 1, \quad R \ge 1. \tag{8}$$

As suggested by the above notation R proves to be the eigenvalue of the Frobenius-Perron equation (3), which is satisfied by (8) with $P_0(x) = 1$ independent of R. As an example for a map conjugated to $f_0(x)$ let us take

$$u(x) = \sin^2\left(\frac{\pi}{2}x\right) \tag{9}$$

which leads to

$$P_g(x) = \frac{1}{\pi\sqrt{x(1-x)}} . \tag{10}$$

When the parameter R in (8) corresponds to the escape rate $\kappa = (m-1)\ln 2$, $m = 1, 2, \ldots$, the resulting transformed map consists of the leftmost and the rightmost branches of the Chebyshev polynomial of order $2m$ with suitable scaling such that they map the unit interval onto itself with $x = 0$ as a fixed point. The case

$m = 1$ represents permanent chaos and (9), (10) give the well-known connection [11] between the logistic and the tent maps in this situation. For $m = 2$ the map reads $g(x) = 16x(1-x)[1-4x(1-x)]$, $x_{m_1} = \frac{(\sqrt{2}-1)}{2\sqrt{2}}$, and is shown in Fig. 2.

When relating symmetric maps by conjugation, $u(x)$ has a symmetry property as well, namely $u(x) = 1-u(1-x)$, and if in addition $P_f(x) = P_f(1-x)$, the conjugation also preserves this symmetry. As a consequence, all the symmetric maps conjugated to $f_0(x)$ have a symmetric conditionally invariant density.

Let us now turn to another transformation which can act upon $f(x)$. It is defined by

$$f(x) = f_0(x) - v(f(x)), \quad 0 \le x \le \frac{1}{2R}, \quad 1 - \frac{1}{2R} \le x \le 1 . \tag{11}$$

The function $v(x)$ is smooth, single valued and fulfills the requirements

$$v(x) = v(1-x); \quad 0 \le x \le 1; \quad v(0) = 0; \quad -1 \le v'(x) \le +1. \tag{12}$$

It can easily be shown that transformation (11) leaves the window invariant. An explicit expression can be given for the transformed inverse function as follows:

$$f_l^{-1}(x) = \frac{x + v(x)}{2R}, \quad f_u^{-1}(x) = 1 - f_l^{-1}(x), \tag{13}$$

where the subscripts refer to the lower and upper branches, respectively. The most significant property of this transformation is that $f(x)$ satisfies the Frobenius-Perron equation with the same escape rate as $f_0(x)$ and with the density

$$P(x) = 1 + v'(x). \tag{14}$$

The change in the density due to the transformation has an odd symmetry (with respect to 1/2). In this sense the v transformation is transverse to conjugation. Since conjugation does not alter the statistical properties, one can expect that via a v transformation they vary most significantly. All maps which possess an absolutely continuous conditionally invariant measure are related to the piecewise linear map $f_0(x)$ by a combination of conjugation and v transformation.

$R = 1$ corresponds to fully developed chaos and the v transformation introduced here is a generalization of that for fully developed chaos defined earlier [12–14].

As an example we take

$$v(x) = dx(1-x), \quad -1 \le d \le 1 \tag{15}$$

leading to the piecewise parabolic map

$$f_{\rm PP}(x) = \frac{1 + d - \sqrt{(1+d)^2 - 8dRx}}{2d}. \tag{16}$$

The condition $v'(x) \ge -1$ arises from the requirement that $f(x)$ has to be single valued. [Note that then $v'(x) \le 1$ comes from the symmetry of $v(x)$.] One gets from (11),

$$|f'(x)| = \frac{2R}{1 + v'(f(x))}, \quad 0 \le x \le \frac{1}{2R}, \quad 1 - \frac{1}{2R} \le x \le 1. \tag{17}$$

The modulus of the derivative is bigger than 1 everywhere in the course of the v transformation if $\kappa > 0$ [and even for $\kappa = 0$, except in the limiting case $v'(0) = 1$ at $x = 0$].

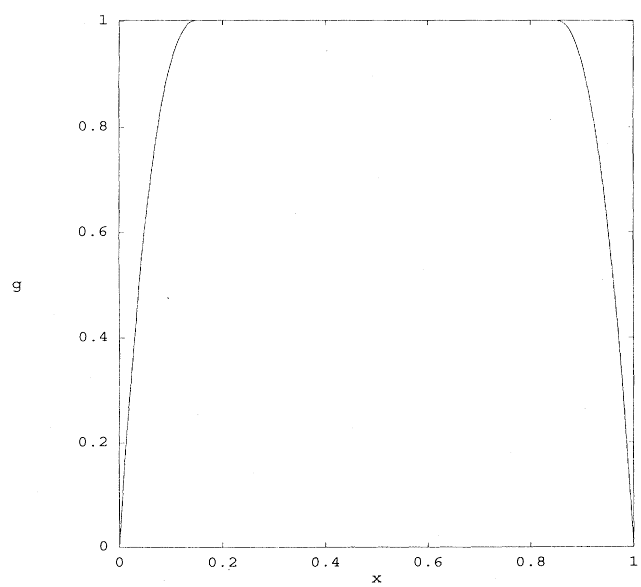

FIG. 2. Map with escape rate $\kappa = \ln 2$, conjugated to the piecewise linear map (8) with $R = 2$.

III. BORDER STATES OF CHAOS

The limiting cases $v'(0) = \pm 1$ represent border lines of transient chaos with an absolutely continuous conditionally invariant measure. Their nature is clearly shown by the behavior of $f'(x)$:

$$v'(0) = 1, \quad f'(0) = R = e^{\kappa}, \quad f'(x_{m_1}) = \infty \tag{18}$$

and

$$v'(0) = -1, \quad f'(0) = \infty, \quad f'(x_{m_1}) = R = e^{\kappa}. \tag{19}$$

In case of map (16), by changing the parameter d, situations (18) and (19) are arrived at when $d = 1$ and $d = -1$, respectively.

Case (18) proves to be the more interesting of the two; therefore we will concentrate on it in the following and comment on (19) only at the end of the paper.

The existence of an absolutely continuous conditionally invariant measure in this limiting situation requires a connection between the properties of the map at $x \approx 0$

and at $x \approx x_m$. In case of the v transformation this is ensured by the even symmetry of $v(x)$ with respect to $x = 1/2$ and leads to the consequence that at the critical state the condition of hyperbolicity is violated, i.e., $|f'(x_m)| = \infty$. In more detail, if

$$v(x) = x + Wx^w + \cdots, \quad w > 1, \quad x \approx 0 \tag{20}$$

then

$$f_l^{-1}(x) = \frac{1}{R}x + \frac{W}{2R}x^w + \cdots, \quad x \approx 0 \tag{21}$$

and

$$f_l^{-1}(x) = \frac{1}{2R} + \frac{W}{2R}(1-x)^w + \cdots, \quad x \approx 1. \tag{22}$$

The order of the maximum of $f(x)$ is $1/w$. The density $P(x)$ approaches 2 when $x \to 0$ according to Eq. (14). One gets then for the natural measure of the leftmost cylinder in the limit of large n

$$\mu^{(n)}(I_0^{(n)}) \approx 2R^n I_0^{(n)} \tag{23}$$

which, since $I^{(n)} \sim [f'(0)]^{-n} = R^{-n}$, tends to a nonzero value. This is in accordance with the general property of the natural measure according to which the crowding index is equal to $1 - \kappa/\lambda$, where λ is the local Lyapunov exponent [15]. The statement following from Eq. (23) is stronger since a zero crowding index would still allow the corresponding measure to tend to zero though slower than a power law. This behavior leads to a phase transition within the framework of the thermodynamic formalism (see Ref. [16] for a review of the thermodynamic formalism). To show it let us introduce in the usual way the partition sum

$$Z_\mu^n(q) = \sum_{i=0}^{2^n-1} [\mu(I_i^n)]^q \tag{24}$$

and the free energy

$$F_\mu(q) = -\frac{1}{q} \lim_{n\to\infty} \frac{1}{n} \ln Z_\mu^n(q), \tag{25}$$

which is closely related to the Rényi entropies $K(q)$ [17,18,14], namely

$$K(q) = \frac{q}{q-1} F_\mu(q), \quad q \neq 1. \tag{26}$$

For $q = 1$ the Rényi entropy becomes the Kolmogorov-Sinai (KS) entropy

$$K \equiv K(1) = -\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{2^n-1} \mu(I_i^n) \ln \mu(I_i^n). \tag{27}$$

In these formulas q plays the role of the inverse temperature. Note that the entropies as defined by (26) and (27) are in fact mean entropy rates.

The existence of a phase transition follows from the upper bound for the Rényi entropy [19,20]

$$K(q) \leq \frac{q}{q-1} \lim_{n\to\infty} \frac{1}{n} \ln \frac{1}{\mu(I_0^n)}, \quad \text{for } q > 1. \tag{28}$$

Since $K(q)$ cannot be negative, (28) and (23) immediately lead to

$$K(q) = 0, \quad \text{for } q > 1. \tag{29}$$

A similar result for permanent chaos in the critical state (corresponding to $R = 1$) has been obtained [19–24], with the essential difference that in permanent chaos $\mu(I_0^{(n)})$ has a power law decay when n tends to infinity. A basic question is the nature of the transition at $q = 1$. We turn now to its discussion.

The starting point is that the natural measure on the repeller develops a δ function singularity at $x = 0$. The strength of this δ function is not determined solely by the leftmost cylinder set but an infinite number of other cylinder sets contribute as well. Consider the cylinder sets at the nth level and then take their mth preimages on the left branch of the map. The total contribution to the strength of the δ function is provided by the quantity

$$M = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=0}^{2^n-1} \mu(f_l^{-m}(I_i^{(n)})). \tag{30}$$

The numerical result for map (16) is shown in Fig. 3. It suggests that at least for large enough R values the strength of the δ function is unity, i.e., the total measure of the rest of the repeller is zero. If this is the case, it can easily be seen that the KS entropy $K(1) = 0$, while $K(q)$

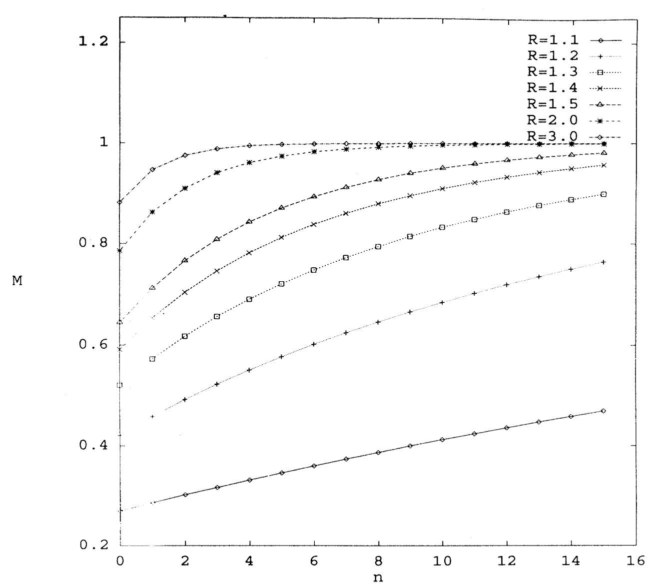

FIG. 3. Strength of the δ function in the natural measure of the map (16) with $d = 1$ for different escape rates as a function of n related to the number of cylinders 2^n covering the repeller.

can be nonvanishing for $q < 1$. Since the free energy is related to the Rényi entropy by Eq. (26) this means that the phase transition is continuous. If this is true for any $R > 1$ then a line of continuous phase transitions is present, which terminates in a discontinuous transition at $R = 1$, where $K(1) > 0$. This behavior can be valid, if the strength of the δ function drops discontinuously to zero when $R \to 1$. Figure 3 can then be interpreted as exhibiting a kind of critical slowing down in the sense that the real strength of the δ function can be observed for larger and larger n values when approaching the limit of permanent chaos, $R = 1$.

A further support of this possibility can be obtained by investigating the Lyapunov exponent. Namely, if the part of the repeller outside $x = 0$ has zero measure, the average Lyapunov exponent is equal to $\kappa = \ln R$ when $R > 1$ while it is known to be 0.5 at $R = 1$ [25,12] which means that a discontinuity shows up in the Lyapunov exponent when $R \to 1$. This tendency is clearly exhibited by the numerical results in Fig. 4 for the map (16) with $d = 1$.

It has to be emphasized that the possibility of $M = 1$ for $R \geq R_c > 1$ but $1 > M > 0$ for $R_c > R > 1$ cannot be excluded according to the numerical data. It is straightforward, however, to adjust the above discussion of the phase transition to this situation if it proves to be the case.

A few words are necessary concerning the other limiting situation (19). In this case there is no qualitative change in the thermodynamic properties when one leaves permanent chaos and enters the territory of transient chaos. Namely, since $f'(0) = \infty$, it can easily be seen that $\mu(I_0^{(n)})$ tends to zero faster than exponentially. From this circumstance $K(q) = \infty$ follows for $q < 0$, in the same way as in the case of permanent chaos [23,13,24].

Finally we should mention that at the limiting situation (18) the generalized dimensions $D(q)$ [17,26,27] exhibit a phase transition like singularity, too. Namely, by partitioning the interval I to uniform boxes the probability of the leftmost box remains finite when the size of the boxes goes to zero. Then it follows by applying an inequality for $D(q)$ [23] that $D(q) = 0$ for $q > 1$. An essential difference, however, as compared to $K(q)$ is that the singularity in $D(q)$ disappears in the limit of permanent chaos, $\kappa \to 0$. On the other hand in both of the limiting cases (18), and (19) a phase transition has been found in $D(q)$ at $q = -1$ in case of permanent chaos [20]. The properties of these phase transitions in the region $\kappa > 0$ would be worth studying.

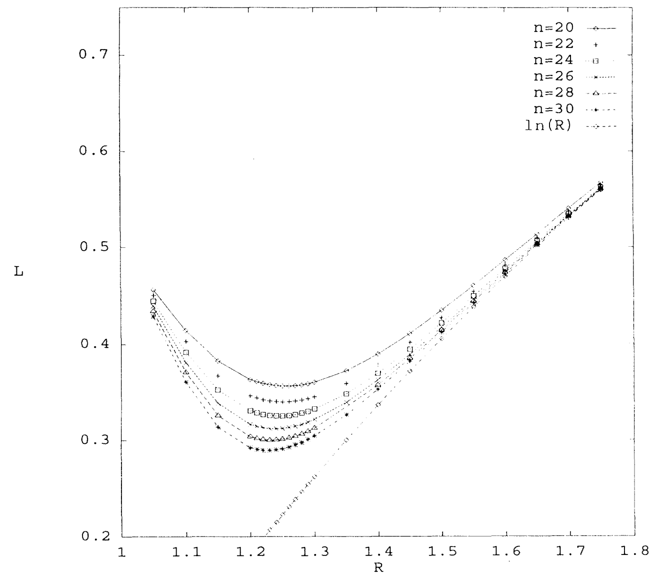

FIG. 4. Lyapunov exponent for the map (16) with $d = 1$ as a function of R related to the escape rate.

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